# Vergleichsstellensätze and Quantum Resource Theories

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# Overview

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# Probability Theory

For a given probability measure $p$ we are interested in questions related to the transformations $T$, where $T$ is a stochastic kernel such that we want it to preserve some zero-resource (free) objects.

In this context $p$ can be mapped to another distribution $q$ asymptotically if there exists $n_0 \in \mathbb{N}$ such that, for any $n \geq n_0$ there is a free transformation $T$ such that $T(p^{\otimes n}) = q^{\otimes n}$. In other words these are free transformations assisted with i.i.d. copies and hence we call them *asymptotic transformations*.

Similarly, if we use a catalyst distribution $r$ (uncorrelated) then we can say say that $p$ can be transformed to $q$ catalytically if there exists a distribution $r$ and a free transformation $T$ such that $T(p \otimes r) = (q \otimes r)$.

# Quantum Resource Theories - A Recap

**QRTs** are information-theoretic models of physical processes defined under a **restricted set of physical operations** i.e certain systems are accessible and only certain operations can be performed.

The permissible operations are called **free** and likewise any state that is permissible is called a **free state**, similarly any state that is not accessible is called a **resource state.**
**Example:** Entanglement, where free operations and states are charachterized by LOCC and entanglement is a resource.

The most basic task in QRT is how one resource can be converted to another using the free operations of the theory.

# Quantum Resource Theories - A Recap

Let's define the set of free operations as $\mathcal{O}(\mathcal{H}^{\mathcal{A}} \rightarrow \mathcal{H}^{\mathcal{B}})$ then for the density matrices $\rho \in \mathcal{S}(A)$ and $\sigma \in \mathcal{S}(B)$ the one-shot convertibility can be defined as the existence of the CPTP map $\Phi \in \mathcal{O}(\mathcal{H}^{\mathcal{A}} \rightarrow \mathcal{H}^{\mathcal{B}})$ such that $\sigma = \Phi(\rho)$.

Similarly, for the map $\Phi$ we can define asymptotic transformations if there is $n_0 \in \mathbb{N}$ such that, for any $n \geq n_0$ the map justifies $\sigma^{\otimes n} = \Phi(\rho^{\otimes n})$, where we abandon the oneshot scenario and consider transforming multiple copies of the same state and finally we say that the transformation consumes a resource state, catalyst, $\omega$ if the transformation is defined by $\sigma \otimes \omega = \Phi(\rho \otimes \omega)$.

# Quantum Resource Theories - A Recap

Clearly these transformations (free operations) induce a preorder (a transitive and reflexive binary relation) on the state space. Since they are both **reflexive** and **transitive**. Hence, in the regime of QRT there is an advantage of studying the preorders themselves as it allows for comparison of resources.

If $\rho \to \sigma$ using any of the convertibility tasks defined earlier then $\rho$ has no less resource than $\sigma$.

The most useful charachterization of the preorder is the majorization preorder.

# Majorization - A Primer

Let us denote by $p^{\downarrow}$ for the decreasing rearrangement of p, i.e.the vector $p^{\downarrow}$ and $p$ have the same coordinates up to permutation. So for probability distributions $p$ and $q$ we say that $p$ majorizes $q$ or $q \prec p$ if and only if:

- (Initial sum condition) for $k = 1, \ldots, d, \sum_{i=1}^{k} p_i^{\downarrow} \geq \sum_{i=1}^{k} q_i^{\downarrow}$ and $\|p\|_1 = \|q\|_1$ or
- (bistochastic matrix condition) there is a bistochastic probability matrix $T = (T_{i,j})_{i,j=1}^{n} \in \mathcal{M}_n(\mathbb{R}_+)$ (i.e., $T$ is an $(n \times n)$-matrix with non-negative entries whose rows and columns sum up to 1) such that $Tp = q$.

# Majorization - A Primer

**Theorem**

*Nielsen's Theorem* $\psi \prec \phi$ *if and only if the schmidt coefficient of* $\psi$ *is majorized by the schmidt coefficient of* $\phi$ *i.e* $x \prec y$

# Current Work - Summary

Given the questions above, in this work we provide a generalization of several previous results using the newly introduced seperation theorems in real semi-algebraic geometry known as Vergleichsstellensätze.

We show that these methods, used in the framework of preordered semirings, can be used to derive conditions for free transformations assisted with both asymptotic and catalytic convertibility. The theoretical nucleus of the results presented here is **[Fritz 2021, parts I, II]**, which provides sufficient conditions for the above asymptotic and catalytic transformations in a variety of settings. Especially, we see that the theory developed in **[Fritz 2021]** yields some known results on majorization.

# Preliminaries

A (commutative) preordered semiring can be defined as the tuple $(S, +, \cdot, 0, 1, \leq)$ where $S$ is a set equipped with

- two binary operations $+$, $\cdot$, which are both commutative and associative
- the neutral elements 0 and 1 exist
- $0a = 0$ and $(x + y) \cdot z = x \cdot z + y \cdot z$ for all $x, y, z \in S$
- the preorder $\leq$ (a transitive and reflexive binary relation) is defined such that for $a \leq b$ implies $a + c \leq b + c$ and $ac \leq bc$ for all $a, b, c \in S$ and $0 \leq 1$.

# Preliminaries

(i) $\mathbb{R}_+ = [0, \infty)$ viewed as a pre-ordered semiring w.r.t. the usual sum and multiplication of real numbers and the usual order,

(ii) $\mathbb{R}_+^{\mathrm{opp}}$, which is the same as above except that the order is reversed,

(iii) $\mathbb{TR}_+ = [0, \infty)$ (the tropical semiring) viewed as a pre-ordered semiring w.r.t. the usual product, the sum $x +' y := \max\{x, y\}$, and the usual order, or

(iv) $\mathbb{TR}_+^{\mathrm{opp}}$, which is the same as above except that the order is reversed.

In the preordered semiring reversing the preorder produces another preordered semiring.

# Preliminaries

An important property for a preordered semiring $S$ is *polynomial growth*. Every element in $S$ must be dominated by the *power universal* $u \in S$.

So, Let $S$ be a preordered semiring.

- A *power universal element* is $u \neq 0$ and $u \in S$ s.t whenever $x \leq y$ for non-zero $x, y \in S$ there is $k \in \mathbb{N}$ such that $y \leq xu^k$.
- A *power universal pair* is $(u_-, u_+)$, i.e., $u_\pm \in S$ are non-zero, $u_- \leq u_+$, and, for any non-zero $x, y \in S$ such that $x \leq y$, there is $k \in \mathbb{N}$ such that $u_-^k y \leq u_+^k y$. Also $u = u_+ u_-^{-1}$
- We say that $S$ is of polynomial growth if it possesses a *power universal pair*

# Preliminaries

Let $S$ be a preordered semiring. The *asymptotic spectrum* denoted by $\Delta(S, \leq)$ is defined as the set of all monotone semiring homomorphisms $f$ (with its usual addition, multiplication and order.) The elements of the asymptotic spectrum are referred to as *spectral points*.

# Vergleichsstellensätze - Theorem 1

Let $S$ be a preordered semiring of polynomial growth and $1 \geq 0$, and let nonzero $x, y \in S$. Then the following are equivalent:

- for all montone homomorphism $\phi : S \to \mathbb{K}$, $\mathbb{K} \in \{\mathbb{R}_+, \mathbb{TR}_+\}$
$$\phi(x) \leq \phi(y)$$
.
- For every $\epsilon > 0$ and all $n \gg 1$
$$x^n \leq u^{\lfloor \epsilon n \rfloor} y^n$$

# Vergleichsstellensätze - Theorem 1

Moreover, given that $\boldsymbol{\phi(x) \leq \phi(y)}$. Then the following also hold:

- $u^k x^n \leq u^k y^n$,
- if, additionally, $y$ is a power universal then

  $$x^n \leq y^n$$

  for sufficiently large $n \in \mathbb{N}$, and
- $xa \leq ya$ where the catalyst can be chosen to be $a = u^k \sum_{k=0}^{n} x^k y^{n-k}$ for some $k \in \mathbb{N}$ where $n \in \mathbb{N}$ is sufficiently large.

# Vergleichsstellensätze - Theorem 1

The theorem requires the assumption that $1 \geq 0$. However, there are scenarios specifically in the setting of preordered semirings in probability and information theory where the assumption for $1 \geq 0$ doesn't hold.

**Example:** Probability measures are normalized to 1, which means that we only want measures of the same normalization to be comparable. In particular, the zero measure will not be comparable to any normalized measure, which results in the case $1 \not\geq 0$ and $1 \not\leq 0$.

Given this we introduce the further two Vergleichsstellensätze introduced in **Fritz2021b.**

# Vergleichsstellensätze - Theorem 2

In order to drop the condition $1 \geq 0$ we need to introduce some new montone homomorphisms that take infinitesimal information encoded in the form of monotone derivations in addition to the monotone homomorphisms to the nonnegative reals and tropical reals.

Let $S$ and $T$ be preordered semirings. We say that the monotone homomorphism is **degenerate** if for all $x, y \in S$,

$$x \leq y \implies \phi(x) = \phi(y).$$

Otherwise $\phi$ is **nondegenerate**.

# Vergleichsstellensätze - Theorem 2

Let $S$ be a semiring and $\phi : S \to \mathbb{R}_+$ a homomorphism. Then a $\phi$-**derivation** is a map $D : S \to \mathbb{R}$ such that the **Leibniz rule**

$$D(xy) = \phi(x)D(y) + D(x)\phi(y)$$

holds for all $x, y \in S$.

# Vergleichsstellensätze - Theorem 2

Let $S$ be a preorder semiring of polynomial growth, with a power universal pair $(u_-, u_+)$ satisfying some additional conditions. and let nonzero $x, y \in S$, $x \sim y$. If, for any monotone homomorphism $\varphi : S \to \mathbb{K}$, $\mathbb{K} \in \{\mathbb{R}_+, \mathbb{R}_+^{\mathrm{op}}, \mathbb{TR}_+, \mathbb{TR}_+^{\mathrm{op}}\}$, we have

- either $\varphi$ is nondegenerate and has a trivial kernel and $\varphi(x) < \varphi(y)$ or
- $\mathbb{K} = \mathbb{R}_+$ and $\varphi$ is degenerate but $D(x) < D(y)$ for any derivation $D$ at $\varphi$ such that $D(u_+) = D(u_-) + 1$,

then there is a nonzero $a \in S$ such that $ax \leq ay$.

# Vergleichsstellensätze - Theorem 3

Let $S$ be a zerosumfree preordered semidomain of polynomial growth with a power universal element $u$ and with a monotone homomorphism $\|\cdot\| : S \to \mathbb{R}_+$ such that

$$x \leq y \quad \Rightarrow \quad \|x\| = \|y\| \quad \Rightarrow \quad x \sim y.$$

Let $x, y \in S \setminus \{0\}$ be such that $\|x\| = \|y\|$. The conditions of Theorem 2 with non-strict inequalities is equivalent with the following statement: for all $\varepsilon > 0$ we have $x^n \leq u^{\lfloor n\varepsilon \rfloor} y^n$ for sufficiently large $n \in \mathbb{N}$. If the conditions (i) and (ii) hold as they are, also the following are true:

- $u^k x^n \leq u^k y^n$,
- if, additionally, $y$ is a power universal then

  $$x^n \leq y^n$$

  for sufficiently large $n \in \mathbb{N}$, and
- $xa \leq ya$ where the catalyst can be chosen to be $a = u^k \sum_{k=0}^{n} x^k y^{n-k}$ for some $k \in \mathbb{N}$ where $n \in \mathbb{N}$ is sufficiently large.

# Vergleichsstellensätze - Theorem 3

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# Klimesh

Let $p$ and $q$ be probability vectors. Suppose that $p$ and $p$ do not both contain components equal to 0 and that $p \neq q$. Then $p$ catalytically majorizes $q$ i.e $p \otimes r \prec q \otimes r$ if and only if $H_\alpha(p) < H_\alpha(q)$ for all $\alpha \in \mathbb{R}$ where $H_\alpha$ are the Rényi entropies defined, for all $p = (p_1, \ldots, p_n)$ through

$$H_\alpha(p) = \begin{cases} \log \sum_{i=1}^{n} p_i^\alpha, & \alpha > 1, \\ \sum_{i=1}^{n} p_i \log p_i, & \alpha = 1, \\ -\log \sum_{i=1}^{n} p_i^\alpha, & 0 < \alpha < 1, \\ -\sum_{i=1}^{n} \log p_i, & \alpha = 0, \\ \log \sum_{i=1}^{n} p_i^\alpha, & \alpha < 0 \end{cases}$$

where $0 \log 0 = 0$ and, if $p_i = 0$ for some $i$, then $H_\alpha(p) = \infty$ for all $\alpha \geq 0$.

# Aubrun and Nechita

For the finite probability vectors $p$ and $q$ the following are all equivalent.

(i) $p^{\otimes n} \prec q^{\otimes n}$

(ii) $p \otimes r \prec q \otimes r$

(iii) $H_\alpha(p) \geq H_\alpha(q) \quad \forall \alpha \geq 1$

# Jensen

Let $p = p^{\downarrow}$ and $q = q^{\downarrow}$ be two probability distributions and assume that

$$H_\alpha(p) > H_\alpha(q) \text{ for all } \alpha \in [0, \infty].$$

then for sufficiently large $n$

$$P^{\otimes n} \prec Q^{\otimes n}$$

# How do we re-derive these results?

- Define different majorization orders
- Show that they are preordered semirings with polynomial growth
- Derive the monotone homomorphisms and derivations
- Use the monotone homomorphisms and derivations for these rings to re-derive the results.

# Additional Majorization Preorders

We define the submajorization preorder as $q \preccurlyeq p$ if, for all $k = 1, \ldots, n$,

$$\sum_{i=1}^{k} q_i^{\downarrow} \geq \sum_{i=1}^{k} p_i^{\downarrow}.$$

or there exists bistochastic map $T = (T_{i,j})_{i,j=1}^{n} \in \mathcal{M}_n(\mathbb{R}_+)$ such that $Tq \leq p$. When equipped with the submajorization order $(S, \preccurlyeq)$ becomes a preordered semiring which we call the *submajorization semiring* and denote by $S_{\mathrm{sm}}$.

# Additional Majorization Preorders

$q \prec p$ if $q \preccurlyeq p$ and $\|p\|_1 = \|q\|_1$ where $\|\cdot\|_1 : S_0 \to \mathbb{R}_+$ is the 1-norm, $\|(p_1, \dots, p_n)\|_1 = p_1 + \cdots + p_n$. When equipped with the majorization order $(S, \prec)$ becomes a preordered semiring which we call the *majorization semiring* and denote by $S_{\mathrm{m}}$.

# Additional Majorization Preorders

$q \preceq p$ if $q \prec p$ and $\|p\|_0 = \|q\|_0$ where $\| \cdot \|_0$ is the '0-norm', i.e., $\|p\|_0$ is the size of the support of $p = (p_1, \ldots, p_n)$, i.e., the number of those $i$ such that $p_i > 0$. When equipped with the modified majorization order $(S, \preceq)$ becomes a preordered semiring which we call the *modified majorization semiring* and denote by $S'_{\mathrm{m}}$.

# Montone Homomorphisms over the Submajorization Semiring

Using **Theorem 1** The monotone homomorphism $\varphi : S_{\mathrm{sm}} \to \mathbb{R}_+$ and $\varphi : S_{\mathrm{sm}} \to \mathbb{TR}_+$ can be characterized respectively as

- $f_\alpha(p) := \sum_{i \in I} p_i^\alpha$ For all $\alpha > 1$
- $F_\alpha(p) := \max_{i \in I} p_i^\alpha$ For all $\alpha = \infty$

# Montone Homomorphisms over the Majorization Semiring

Similarly, The monotone homomorphism $\varphi : S_{\mathrm{m}} \to \mathbb{R}_+$ and $\varphi : S_{\mathrm{m}} \to \mathbb{R}_+^{op}$ can be characterized as

- $f_\alpha(p) := \sum_{i \in I} p_i^\alpha$ For all $\alpha \in [0, 1]$
- $f_\alpha(p) := \sum_{i \in I} p_i^\alpha$ For all $\alpha > 1$

Moreover, the monotone homomorphism $\varphi : S_{\mathrm{sm}} \to \mathbb{TR}_+$ and $\varphi : S_{\mathrm{sm}} \to \mathbb{TR}_+^{op}$ can be characterized as

- $F_\alpha(p) := \max_{i \in I} p_i^\alpha$ For all $\alpha = \infty$

# Montone Homomorphisms over the Majorization Semiring

In addition to the monotone homomorphisms the condition for $\|p\|_1 = \|q\|_1$ creates a degeneracy.

Hence, The only additive monotone derivation $D : S_{\mathrm{m}} \to \mathbb{R}$ at $\|\cdot\|_1$ is the Shannon entropy $H_1$:

$$H_1(p_1, \ldots, p_n) = -\sum_{i=1}^{n} p_i \log p_i$$

# Montone Homomorphisms over the Modified Majorization Semiring

In addition to the monotone homomorphisms the condition for $\|p\|_1 = \|q\|_1$ creates a degeneracy.

The monotone homomorphism are exactly the same as those for the majorization semiring with the additional condition that

$$f_\alpha(p) := \sum_{i \in I} p_i^\alpha \text{ For all } \alpha < 0$$

# Montone Homomorphisms over the Modified Majorization Semiring

However, now we have degeneracy at two points.

Hence for $\varepsilon \in (0, 1/2)$, the derivative map at $D_1 = cH_1$ where $c_1 = \big(1 - h_1(\varepsilon)\big)^{-1} > 0$ where, in turn, $h_1(x) = -x\log x - (1-x)\log(1-x)$.

And the additive monotone derivation $D_0 : S'_{\mathrm{m}} \to \mathbb{R} \,\|\cdot\|_0$ is defined as

$$D_0 = c_0 H_0$$

where $c_0 = -\big(2 + h_0(\varepsilon)\big)^{-1} > 0$ where, in turn, $h_0(x) = \log(x - x^2)$ for all $x \in (0, 1)$

# Re-deriving the classical results

The result due to Jensen is a direct consequence of the monotone homomorphism of the $\mathsf{S}_{\mathrm{m}}$.

The result due to **Aubrun and Nechita** follows from the monotone homomorphism of the $\mathsf{S}_{\mathrm{sm}}$.

Lastly, the result from **Klimesh** can be re-derived from the monotone homomorphism of the $\mathsf{S}_{\mathrm{m}} and S'_{\mathrm{m}}$.

# The End